ON THE DISTRIBUTION OF ATMOSPHERIC SODIUM

BY

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§ 1. THE SODIUM D-LINES IN NIGHT SKY LUMINESCENCE.

Since the discovery by Slipher (1) of a yellow radiation in the night sky spectrum and its subsequent identification as the sodium D-lines (Bernard (2, 3), Cabannes, Dufay & Gauzit (4, 5), Vegard & Tønsberg (6)) several investigators have attacked the problem of locating this radiating sodium.

Early measurements by Garrique (7) showed that the brightness B_ζ of the sodium light in the night sky luminiscence increased with increasing zenith distance ζ, thus establishing its connection with the earth's atmosphere.

Cabannes et al. (1. c) assumed that the light-emitting sodium in the atmosphere was limited to a fairly thin layer uniformly distributed over large geographical areas. The intensity of light from such a luminiscent layer will vary according to a simple function of the zenith distance. By means of Garrigue's intensity measurements and after having made allowances for atmospheric extinction and for light diffused into the observation direction from other parts of the sky, Cabannes et al. (1. c) estimated the height (H) of the sodium layer to be about 130 km above the surface of the earth.

Some time later Dufay and Tcheng (8, 9) from new series of measurements of the intensity variation of the sodium light in the night sky, calculated the height by the same procedure. They give the value $H = (80 \pm 20)$ km. This seems to be in rather poor agreement with the results of Cabannes and collaborators. As Dufay and Tcheng, however, have pointed out, the incertitude in both the measured B_ζ-values as in the applied corrections are so great that until new precise and longer series of measurements are obtained, no better estimate of the hight can be given by this method.

Table I.
Mean Hight of the Radiating Sodium-layer as Derived from Night Sky Observations.

Authors	Place	Height, km
Cabannes, Dufay, Gauzit (4)	Pic du Midi	130
Dufay, Tcheng (9)	Haute-Provence	80 ± 20

§ 2. TWILIGHT OBSERVATIONS.

The enhancement of the yellow radiation in the twilight was probably first observed by Currie & Edwards (10), and Vegard & Tønsberg (11), and shortly afterwards studied in more detail by Bernard (2) at Tromsø. Observing towards zenith he found that the intensity of the sodium light remained constant during the earlier part of the twilight and then dropped fairly abruptly down to a small value, when the sun had gained a certain depression below the horizon.

The enhancement obviously was connected with the radiation from the sun in a very direct manner. Bernard assumed that the sodium light in the twilight was a resonance radiation and thus excited directly by a part of the visible sunlight.

At the instant of "disappearance" of the D-line the border between sunlit atmosphere and atmosphere in shadow in the zenith direction lay about 60 km above the surface of the earth (atmospheric refraction taken into account). This should then be the upper limit of the radiating sodium.

Elwey and Farnsworth (12) have studied the enhancement of the D-lines at Mont Locke, Texas. Contrary to Bernard they find the intensity to fall constantly during twilight, following an exponential law, and dropping from 1 to 0.01 (arbitraty scale) when the earth's shadow rises from 75–115 km.

Table IIa.

Upper Limit of Radiating Sodium Derived from Twilight Observations by the Method of Bernard.

Authors	Place	Heights, km
Bernard (2)	Tromsø St. Auban	60
Elwey, Farnsworth (12)	Mount Locke	70–115*)

*) (Corrected for atmospheric refraction ?)

Bernard's very direct method is based on the assumption that the earth should be the opaque matter which cuts out the exciting radiation.

Vegard in collaboration with Tønsberg and Kvifte (6, 13) adopted a method by which they determined the instant of disappearance of the sodiumlight for two points with different zenith angles ($\zeta_1 = 0°$, $\zeta_2 = 80°$) in the azimuthplane of the sun. If the earth were the effective screen for the exciting light, the shadow border should intersect the direction of observation in the two cases at the same height, H_u, above the earth. In fact the observation with the greater zenith-angle gives the greater height. Now introducing a parameter, H_s, with the physical meaning of effective screen height for the active light, it is possible from the two observations to determine that value of H_s which will give the same H_u in both cases. The only supposition here is then that *the upper limit of the radiating sodium* (which shows the enhancement effect) *lies in the same height above the earth at different geographical positions.*

The method of Vegard and collaborators has also been tried by Dufay (14) and Penndorf (15, 16), the latter having recalculated some of the data from the publication of Vegard and Tønsberg in a slightly different manner. Table II b contains the results obtained as yet by this method.

Table II b.

Upper Limit, H_u, of Radiating Sodium and Screening Height, H_s, of Activating Radiation as Derived from Twilight Observations.

Authors	Place	Heights, km	
		H_u	H_s
Vegard, Tønsberg (6)	Tromsø Oslo	109 119	50 58
Vegard, Kvifte (13)	Oslo	105	44
Dufay (14)	?	96	25
Penndorf (15)	Tromsø Oslo	94 108	34 47

It is, perhaps, now appropriate to point out the difference between the heights given in the tables I and II. The data in table I relate to a *mean height* of a supposed sodiumlayer which radiates in nighttime *unaffected by direct sunlight,* whereas the heights in table II a and H_u in table II b should give the *upper limit* of the sodium in the atmosphere which is *excited by direct sunlight* in some way. Since the sodium-radiations from the nightsky and the twilight luminiscense are excited in different manners, there need not necessarily be any connection between the two phenomena (Vegard and Tønsberg (6)). If, however, a connection exists, the data in table I should be less than those in the tables II a and II b. The height given by Dufay and Tcheng might, together with the values in table II b, indicate such a connection.

The last column in table II b is of special interest. If the screening height, H_s, is greater than say 40 km as the majority of figures indicates, the screening effect may as suggested by Vegard et al. be attributed to ozon absorption, and the exciting radiation is then some sort of ultraviolet light.

If on the other hand the height obtained by Dufay is correct, it is more likely that the enhancement is a resonance effect. By calculations of the absorption of the D_1-D_2-lines in solar radiation by the lower atmosphere, Bricard and Kastler (17) mean to show that the atmosphere up to 25 km will act as a comparatively effective screen for this radiation.

An analysis of the method might now be suitable to the purpose of gaining information of the obtainable precision of the measured heights, information which might give reasons for discerning between the two alternatives of excitation.

§ 3. GENERAL THEORY OF THE EVALUATION OF H_u AND H_s.

As the area over which the observations are carried out is of comparatively small extension, it is a sufficient good approximation to suppose that the earth is a sphere with a radius, R, equal to the distance from the observation spot to the center. The layer of atmosphere which acts as a screen for the active solar light, and that which contains the radiating sodium in twilight, must then be bounded by spherical surfaces concentrical with the earth. The radii are $R_s = R + H_s$ and $R_u = R + H_u$ respectively.

The sphere in fig. 1 is the outer limit of the radiating sodium in twilight. C is the common center of the spheres mentioned. O is the spot of observation on the surface of the earth and

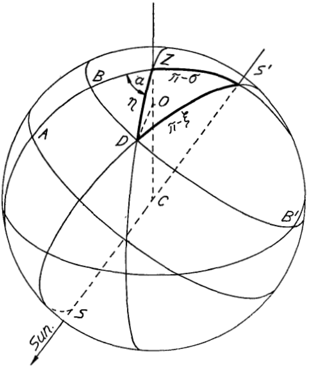

Fig. 1.

OZ its zenith direction. The position of the sun is given by the centerline SS'.

The borderline between the sunlit part of the sphere and the part shadowed by the screening layer is the circle BDB' in a plane perpendicular to the direction SS'. ξ is the angle between SS' and a line from the center C to this circle.

An observation made in the direction OD, fixed by the zenith distance ζ and the azimuth-angle (a) measured from the vertical plane through the spot of observation and the sun, will register the disappearance of the enhanced sodiumlight just as the borderline BB' intersects the direction of observation (in D). The zenith-angle of the sun may then be σ.

At such an instant the spherical triangle $S'ZD$ will give the relation:

$$\cos \xi = \cos \eta \cos \sigma + \sin \eta \sin \sigma \cos a \qquad (1)$$

η is the angle between the radii CZ and CD.

The two vertical planes through the spot of observation: COD containing the direction of observation, and CSZ through the sun, are shown in fig. 2 and fig. 3 respectively.

The triangle COD in fig. 2 furnishes an equation between the unknown angle η and the measured zenith angle ζ:

$$\sin (\zeta - \eta) = \frac{R}{R_u} \sin \zeta = \frac{1}{q_u} \sin \zeta \qquad (2)$$

$$q_u = \frac{R_u}{R} = \left(1 + \frac{H_u}{R}\right) \qquad (3)$$

In fig. 3 EB is a solar ray tangent to the upper limit of the screening layer, B the intersection point between this ray and the upper border of the sodium "layer".

From triangle CBE:

$$\sin \xi = \frac{R_s}{R_u} = \frac{q_s}{q_u} \qquad (4)$$

$$q_s = \frac{R_s}{R} = \left(1 + \frac{H_s}{R}\right) \qquad (5)$$

Elimination of η from eq. (1) by means of eq. (2) and the relation

$$\sin^2 \eta + \cos^2 \eta = 1$$

gives:

$$\cos \xi = \frac{1}{q_u}\left(- a \sin \zeta + \beta \left(q_u^2 - \sin^2 \zeta\right)^{\frac{1}{2}}\right) \qquad (6)$$

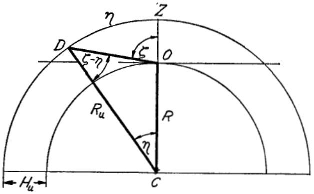

Fig 2.

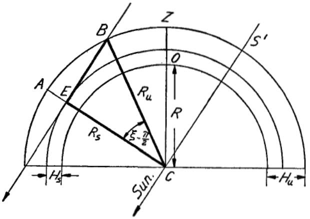

Fig. 3.

with the abbreviations:

$$\left.\begin{aligned} \alpha &= \sin\sigma\cos\zeta\cos a - \cos\sigma\sin\zeta \\ \beta &= \cos\sigma\cos\zeta + \sin\sigma\sin\zeta\cos a \\ \gamma &= \sin\sigma\sin a \end{aligned}\right\} \quad (7)$$

γ having been added for later purposes. α, β and γ are related by

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \quad (8)$$

If now the thickness H_s of the absorbing layer and the height H_u to the upper limit of the radiating sodium do not undergo any variation during some period of time, sufficiently long to determine the instance of disappearance of the enhanced sodiumlight for two sets of the angles (a) and ζ, then the quantities q_s, q_u and ξ (eq. (5). (3) and (4)) will remain constant during this period, and q_u may be determined by means of eq. (6) in the following manner:

$$\left.\begin{aligned} q_u\cos\xi &= -\alpha_1\sin\zeta_1 + \beta_1(q_u^2 - \sin^2\zeta_1)^{\frac{1}{2}} \\ &= -\alpha_2\sin\zeta_2 + \beta_2(q_u^2 - \sin^2\zeta_2)^{\frac{1}{2}} \end{aligned}\right\} \quad (6\,a)$$

The indices 1 and 2 refer to the two sets of observations. Eq. (6a) leads to a biquadratic equation in q_u with the solution:

$$\left.\begin{aligned} q_u = \frac{\pm 1}{(\beta_1^2 - \beta_2^2)}&[(\beta_1^2-\beta_2^2)(\beta_1^2\sin^2\zeta_1 - \beta_2^2\sin^2\zeta_2) \\ &+ (\beta_1^2+\beta_2^2)A^2 \pm 2\,\beta_1\beta_2 A\,(A^2 + \\ &+ (\beta_1^2-\beta_2^2)\,(\sin^2\zeta_1 - \sin^2\zeta_2))^{\frac{1}{2}}]^{\frac{1}{2}} \end{aligned}\right\} \quad (9)$$

A being given by:

$$A = (\alpha_1\sin\zeta_1 - \alpha_2\sin\zeta_2) \quad (10)$$

The combination of eq.s (3) and (9) finally gives H_u:

$$H_u = R(q_u - 1) \quad (11)$$

The expression for H_s is obtained in a similar way. It is convenient to eliminate q_u from the equations (4) and (6) which results in:

$$\operatorname{cotg}\xi = \frac{1}{q_s\varepsilon}[-\alpha\sin\zeta + \beta(q_s^2\varepsilon - \gamma^2\sin^2\zeta)^{\frac{1}{2}}] \quad (12)$$

Proceeding as before with two sets of observations, the resulting expression for q_s will be:

$$\left.\begin{aligned} q_s = \frac{\pm 1}{\varepsilon_1\varepsilon_2(\beta_1^2-\beta_2^2)}&[(\beta_1^2-\beta_2^2)(\beta_1^2\varepsilon_2^2\gamma_1^2\sin^2\zeta_1 \\ &- \beta_2^2\varepsilon_1^2\gamma_2^2\sin^2\zeta_2) + B^2(\beta_1^2\varepsilon_2 + \beta_2^2\varepsilon_1) \\ &\pm 2\beta_1\beta_2 B(\varepsilon_1\varepsilon_2(B^2 + (\beta_1^2-\beta_2^2)(\gamma_1\varepsilon_2\sin^2\zeta_1 \\ &- \gamma_2^2\varepsilon_1\sin^2\zeta_2)))^{\frac{1}{2}}]^{\frac{1}{2}} \end{aligned}\right\} \quad (13)$$

the additional abbreviations being:

$$\varepsilon = (1 - \beta^2) \quad (14)$$

$$B = (\alpha_1\varepsilon_2\sin\zeta_1 - \alpha_2\varepsilon_1\sin\zeta_2) \quad (15)$$

H_s is then found by eq. (4):

$$H_s = R(q_s - 1) \quad (16)$$

The choice of signs in eqs. (9) and (13) is dependent on which of the two sets of observations, indicated 1 and 2, has the greater zenith distance ζ (reckoned positive towards the sun). q_u and q_s must of course always be positive, and the two terms in the parenthesis should ordinarily be subtracted.

§ 4. SPECIALISATION OF THE OBSERVATIONAL DIRECTION.

The preceeding formulae may be simplified considerably by a suitable choice of the direction of observation. The following cases should be of special interest:

1. One set of observations towards zenith, say $\zeta_1 = 0$, then

$$\left.\begin{aligned} \xi &= \sigma_1 \text{ (from eq. (6)} \\ \alpha_1 &= \sin\sigma_1 \cos a_1 \\ \beta_1 &= \cos\sigma_1 \\ \gamma_1 &= \sin\sigma_1 \sin a_1 \end{aligned}\right\} \quad (7\text{ a})$$

$$q_s = q_u \sin\xi = q_u \sin\sigma_1 \quad (4\text{ a})$$

and

$$\left.\begin{aligned} q_u = \frac{\sin\zeta_2}{(\beta_2{}^2 - \cos^2\sigma_1)} \Big[\alpha_2 \cos\sigma_1 + \\ + \beta_2 \sin\sigma_1 \left(1 - \left(\frac{\gamma_2}{\sin\sigma_1}\right)^2\right)^{\frac{1}{2}} \Big] \end{aligned}\right\} \quad (9\text{ a})$$

An approximation to this expression is obtained by introducing α_2 and β_2 from eqs. (7) and (7a), expanding the square root parenthesis in binominal series $\left(\left|\frac{\gamma_2}{\sin\sigma_1}\right|\right.$ being less than unity for values of a at least up to 80°) and performing the division.

The first approximation, which as well is the exact expression for q_u when $a_2 = 0$, that is when the second series of observations is carried out in the azimuthplane of the sun, is given by:

$$q_{ou} = \frac{\sin\zeta_2}{\sin(\sigma_1 - \sigma_2 + \zeta_2)} \quad (9\text{ b})$$

Combination of this expression with eqs. (11), (4a) and (16) gives further:

$$H_{ou} = R\left(\frac{\sin\zeta_2}{\sin(\sigma_1 - \sigma_2 + \zeta_2)} - 1\right) \quad (11\text{ a})$$

$$H_{os} = R\left(\frac{\sin\zeta_2 \sin\sigma_1}{\sin(\sigma_1 - \sigma_2 + \zeta_2)} - 1\right) \quad (16\text{ a})$$

These are the expressions equivalent to those used by Vegard and Kvifte (13) who have, however, given them in the form:

$$H'_u = \frac{R + H_s}{\cos h_z} - R$$

$$H'_s = R \frac{2\cos h_z \sin\frac{h_a}{2} \sin\left(a_o + \frac{h_a}{2}\right) - \sin h_z \sin(a_o + h_a)}{\cos(a_o + h_a - h_z)}$$

where a_0 denotes the elevation of the direction of the observation undertaken near the horizon, i. e.

$$a_o = \frac{\pi}{2} - \zeta_2$$

and h is the depression of the sun below the horizon, i.e.

$$h_z = \sigma_1 - \frac{\pi}{2}$$

$$h_a = \sigma_2 - \frac{\pi}{2}$$

The expression for H'_u and H'_s may be reduced to

$$H'_u = R\left(\frac{\cos a_o}{\cos(a_o + h_a - h_z)} - 1\right)$$

$$H'_s = R\left(\frac{\cos a_o \cos h_z}{\cos(a_o + h_a - h_z)} - 1\right)$$

which by change of the notations are identical with eqs. (11 a) and (16 a).

The second approximation to q_u in eq. (9a) is:

$$q_{1u} = \frac{\sin\zeta_2}{\sin(\sigma_1 - \sigma_2 + \zeta_2)} \left[1 + \frac{2\sin\sigma_2 \sin(\sigma_2 - \sigma_1)}{\sin\sigma_1 \operatorname{tg}(\sigma_1 - \sigma_2 + \zeta_2)} \sin^2\frac{a_2}{2}\right] \quad (9\text{ c})$$

which is certainly more convenient in use than (9a) because the evaluation of α_2 and β_2 will prove rather labourous when several series of observations shall be handled.

The value of q_s is given to the same approximation by eqs. (9c) and (4a).

The curves in fig. 4 are plots against $\pm a$ of the difference $\Delta H_a = H_a - H_o$, where H_o is the height calculated from the first approximation as given in eq. (11a) and H_a the one obtained

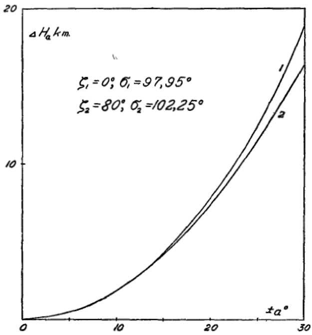

Fig. 4.

from either the exact formula (9a) in combination with eq. (11) (curve 1) or the second approximation to q_u given in eq. (9c) in combination with eq. (11) (curve 2). The numerical data of ζ and σ are mean values taken from the publication of Vegard and Kvifte (see table IV later on). The data relate strictly only to observations in the azimuthplane of the sun, but are here used as if they were the data obtained from observations undertaken in a vertical plane forming an angle "a" with the sun's azimuthplane.

It is at once evident that the first approximation is only fit for use when a is less than say 4°. Otherwise the error in H will exceed 0.3 km.

The second approximation will give practically exact results when $|a|$ is less than 15° and errors not exceeding 0.3 km in an additional interval of 5°.

2. Both sets of observations are carried out in the azimuthplane of the sun.

Then
$$a_1 = a_2 = 0$$
$$\alpha = \sin(\sigma - \varsigma)$$
$$\beta = \cos(\sigma - \varsigma)$$
$$\gamma = 0 \qquad (7\text{b})$$

and eq.s (9) and (13) will be reduced to:

$$q_u = \pm \frac{(\sin^2\zeta_1 - 2\sin\zeta_1\sin\zeta_2\cos(\sigma_1-\zeta_1-\sigma_2+\zeta_2)+\sin^2\zeta_2)^{\frac{1}{2}}}{\sin(\sigma_1-\zeta_1-\sigma_2+\zeta_2)} \qquad (9\text{d})$$

and

$$q_s = \pm \frac{\sin(\sigma_1-\zeta_1)\sin\zeta_2 - \sin(\sigma_2-\zeta_2)\sin\zeta_1}{\sin(\sigma_1-\zeta_1-\sigma_2+\zeta_2)} \qquad (13\text{a})$$

These expressions are not particularly easy in use, especially not eq. (9d) with a square root. It will pay, in planning greater series of observations, to specialise further, e.g. to choose one of the directions of observations towards zenith. Expressions like (9d) and (13a) should be preserved for particular occations when other specialisations are impracticable.

3. A set of simple formulae may be obtained when the two directions of observations in their respective vertical planes are symmetric with respect to the zenith point, i.e. when $\zeta_1 = -\zeta_2$. Putting these values of ζ into eq.s (9d) and (13a), the resulting expressions:

$$q_u{}^\circ = \frac{\sin\zeta_1}{\left(\sin\zeta_1 - \frac{\sigma_1-\sigma_2}{2}\right)} \qquad (9\text{e})$$

$$q_s{}^\circ = \frac{\sin\zeta_1 \sin\left(\frac{\sigma_1+\sigma_2}{2}\right)}{\sin\left(\zeta_1 - \frac{\sigma_1-\sigma_2}{2}\right)} \qquad (13\text{b})$$

will give the formulae to be applied when the observations are carried out symmetrically in the azimuthplane of the sun.

Eq.s (9e) and (13b) are the first approximations to the more general case of symmetric observations when the verticalplanes of the observations do not coincide with the sun's azimuthplane. The exact formulae are then:

$$q_u = \sin\zeta\left[1 + \left(\frac{\alpha_1+\alpha_2}{\beta_1-\beta_2}\right)^2\right]^{\frac{1}{2}} \qquad (9\text{f})$$

$$q_s = \sin\zeta\left[1 + \left(\frac{\alpha_1+\alpha_2}{\beta_1-\beta_2}\right)^2 - \left(\frac{\alpha_1\beta_2+\alpha_2\beta_1}{\beta_1-\beta_2}\right)^2\right]^{\frac{1}{2}} \qquad (13\text{c})$$

(reductions of eq.s (9) and (13) with $\zeta = |\zeta_1|$).

Second approximations to the expressions (9f) and (13c) are

$$q'_u = \frac{\sin\zeta_1}{\sin\left(\zeta_1 - \frac{\sigma_1-\sigma_2}{2}\right)}\left[1 + \frac{\operatorname{cotg}\left(\zeta_1 - \frac{\sigma_1-\sigma_2}{2}\right)\sin\frac{\sigma_1-\sigma_2}{2}}{\sin\frac{\sigma_1+\sigma_2}{2}}\left(\sin\sigma_1\sin^2\frac{a_1}{2} + \sin\sigma_2\sin^2\frac{a_2}{2}\right)\right] \qquad (9\text{g})$$

and

$$q'_s = q'_u \sin\frac{\sigma_1+\sigma_2}{2} \qquad (13\text{d})$$

which should give good results when a_1 and a_2 are not too great.

This mode of observing has been seldom, if at all, applied. It is probably due to the difficulty in obtaining spectrograms fit for intensity measurements in directions with negative zenith-angles (towards the east sky in the evening). Diffused light from the illuminated part of the sky will mask the sodium light in the spectrum. As a great part of this diffused light is plane polarised, it is possible, as proposed by Bricard and Castler (17), to remove some of it by the use of polarisers.

Vervarslinga på Vestlandet

§ 5. THE DETERMINATION OF σ.

The value of the sun's zenith distance, σ, in the preceeding formulae, is the one corresponding to the point of time, τ, when the sodiumlight in the direction of observation is disappearing, that is dropping down to the night-time intensity.

This instant of disappearance is determined by means of a series of short exposure spectrograms taken in the desired direction with a spectrograph of high lightpower. The last spectrogram containing the sodium D-lines will give τ within the timelimits of the exposure interval. Usually this will not, however, be sufficiently accurate for precission measurements, and Vegard and Kvifte (13) have therefore described a method which ought to be more exact. They extrapolate the value of τ from the intensity-time-curve which they construct from the series of spectrograms of the D-lines in twilight.

When thus the τ-value (in M.E.T. reckoned from noon) is determined, the hour angle, t, of the sun is given by

$$t = 15(\tau - 1 + \Delta) + \lambda \qquad (17)$$

and the zenith distance, σ, of the sun by:

$$\cos\sigma = \sin\delta\sin\varphi + \cos\delta\cos\varphi\cos t \qquad (18)$$

The equation of time, Δ, and the declination, δ, of the sun are found from say the "Nautical Almanac". λ and φ are the longitude (E.Gr.) and latitude respectively of the spot of observation.

The error $\Delta\sigma$ in the zenith distance of the sun caused by an error $\Delta\tau$ in the determination of the instance of disappearance of the sodium light in twilight, is found from eq. (18) in combination with eq. (17) by partial derivation of σ with respect to τ. The expression for $\Delta\sigma$ will contain $\sin\sigma$ which may without any appreciable error be put equal to 1 (σ differing at most 13° from $\frac{\pi}{2}$ for observational directions with zenith angles up to 80°).

When expressing $\Delta\tau$ in time-minutes, then $\Delta\sigma$ in angle-degrees will be:

$$\Delta\sigma = 0.25\cos\varphi\cos\delta\sin t\,\Delta\tau \qquad (19)$$

The curves in fig. 5 show the seasonal variation of $\frac{\Delta\sigma}{\Delta\tau}$ deg/min for geographical latitudes

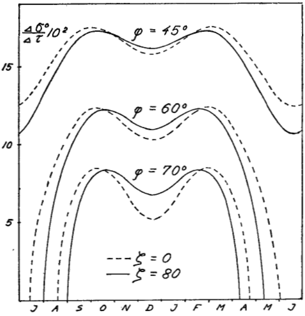

Fig. 5.

45°, 60°, and 70° N of the spot of observation. The pairs of curves correspond to zenith distances σ equal to 97.95° and 102.25°, which are the mean values of the available data of the position of the sun when the sodium light is just disappearing in directions with zenith distances $\zeta_1 = 0°$ and $\zeta_2 = 80°$ respectively.

It is evident from the curves that a spot of observation of high latitude is preferable to one with a lower one. At a time about midway between winter-solstice and one of the equinoxes, the ratio between $\frac{\Delta\sigma}{\Delta\tau}$ at 45° N (South of France) and at 70° N (Tromsø, Norway) is as 2 to 1. At other times of the year the ratio is considerably more in favour of the higher latitude.

The most advantageous time of observation will be around the solstices when the $\frac{\Delta\sigma}{\Delta\tau}$-curves have their minima. At latitudes greater than say 55° N the enhanced sodiumlight will not disappear at all at night during some period of time in the summer for directions of observations with zenithangles $\zeta \geqslant 80°$ (towards the sun). Above say 59° N lat. this will be the case for all directions of observations with $\zeta \geqslant 0$. It is then of course impossible to make any observations. But during the periods in springtime just preceeding (or in autumn just succeeding) this occurrence the conditions are very favourable for a precise determination of σ.

§ 6. DISCUSSION OF SOME RESULTS.

In table II were reviewed the results published as yet of the upper limit and the screening height of the enhanced sodium light in twilight obtained by observing the disappearance of the enhancement. None of the publications contains any complete discussion of the probable errors in the measured heights, and sufficient detailed data for such a discussion are only given in the papers of Vegard et al. (6, 13). These authors have calculated the heights from mean values of ζ and σ obtained from several series of observations. This manner of proceeding, however, gives a poor survey of the dispersion in the material. This is the reason why I have recalculated the heights from data in these two papers. The formulae applied are eq.s (11a) and (16a). They require a complete set of observations, i.e. two observational series with different zenith angles (ζ_1, ζ_2). Some days with single sets had therefore to be excluded.

The results are given in table III and IV.

There is a fairly marked difference between the results in the two tables. The heights in table III are distributed over a range around the mean of $\pm$ 10 km to $\pm$ 12 km in the Oslo- and Tromsø-material respectively. In table IV the maximum error, with a single exception, is about $\pm$ 6 km. The standard deviations for the random errors of the heights show the same difference in precision.

It seems probable that this difference may be attributed to the different methods used in the two cases for determining the instant τ of disappearance of the sodiumlight in twilight. Vegard and Tønsberg (table III) have estimated τ from the last spectrogram containing the D-lines, whereas Vegard and Kvifte have found the τ-value from the D-line time-intensity curve by extrapolation. Evidently this gives the better values.

Denoting by $\Delta_\sigma H$ the error in H caused by an error $\Delta\sigma$ given by eq. (19), partial derivation of eq.s (11 a) and (16 a) with respect to σ gives:

$$\left.\begin{aligned} -\Delta_{\sigma_1} H_u = \Delta_{\sigma_2} H_u = \Delta_{\sigma_2} H_s = 28.5 \frac{\Delta\sigma}{\Delta\tau}\Delta\tau \text{ km} \\ \Delta_{\sigma_1} H_s = -44 \frac{\Delta\sigma_1}{\Delta\tau_1}\Delta\tau_1 \text{ km} \end{aligned}\right\} \quad (20)$$

The constants apply to $\zeta_1 = 0°$, $\zeta_2 = 80°$ and mean values of σ_1 and σ_2 taken from table IV.

Table III.

Heights Calculated from Data in the Paper of Vegard and Tønsberg.

Place	Date 1939	σ_1	ζ_2	σ_2	Heights in km	
					H_s	H_u
Tromsø	28.2	97.80°	80°	102.07°	43	103
	2.3	97.90	,,	102.22	43	104
	8.3	98.13	,,	101.80	19	86
	24.3	97.95	,,	101.77	28	90
	25.3	98.00	,,	101.53	20	82
Means		97.96°	80°	101.88	30.6	93
Standard deviations					11.9	10
Oslo	25.4 *)	97.72°	68°	100.05°	53	111
	,,	97.94	78	101.69	42	104
	26.4	97.97	77	101.75	50	112
	28.4	97.90	80	101.95	35	97
Means		97.88°			45	106
Standard deviations					8.1	7

*) In the morning.

Table IV.
Heights Calculated from Data in the Papers of Vegard and Kvifte.

Dato	σ_1	ζ_2	σ_2	Heights in km	
				H_s	H_u
17.10.42	98.05°	80°	102.35°	40	104
19.10.	98.00	,,	102.37	43	106
16.11.	97.78	,,	102.20	47.5	107
30.11.	98.00	,,	102.35	42	105
2.12.	98.10	,,	102.33	37	102
30.12.	97.82	,,	102.33	49.5	110
8. 1.43	97.95	,,	102.30	43	105
13. 2.	97.60	,,	102.45	61	118
18. 2.	98.05	,,	102.22	36.5	100
2. 3.	97.75	,,	102.12	46.5	106
3. 3.	97.98	,,	102.15	37.5	100
9. 3.	97.92	,,	102.12	39	101
11. 3.	97.75	,,	102.22	49.5	109
12. 3.	97.98	,,	102.22	39.5	102
19. 3.	98.03	,,	102.30	39.5	103
20. 3.	98.05	,,	102.30	38.5	102
23. 3.	97.98	,,	102.20	39	102
24. 3.	97.85	,,	102.20	44.5	105
1. 4.	97.95	,,	102.12	38	100
15. 4.	98.00	,,	102.22	38.5	101
Means:	97.93		102.25	42.5	104.5
Standard deviations:				(6.0) 4.3*)	(4.3) 3.1*)

*) Data from the 13th of February excluded.

$\frac{\Delta\sigma}{\Delta\tau}$ will according to the preceeding section vary with the season and the geographical position of the spot of observation. The data in table IV are collected at Oslo (59.95° N lat.) during the period 17th of October to 15th of April. According to fig. 5 $\frac{\Delta\sigma}{\Delta\tau}$ will then have values ranging from 0.125 to 0.085. Using an average value, the total error in H caused by incorrectness of τ becomes:

$$\Delta_\sigma H_u = 3(\Delta\tau_2 - \Delta\tau_1) \text{ km}$$
$$\Delta_\sigma H_s = 3(\Delta\tau_2 - 1.5\,\Delta\tau_1) \text{ km} \qquad (21)$$

$\Delta\tau$ is measured in time-minutes.

In the most unfavourable case when the errors in τ_1 and τ_2 (then having opposite signs) act in the same direction on $\Delta_\sigma H$, an incorrectness of one and a half minute in each of them suffices to account for the maximum errors of the heights in table III, one minute for those in table IV. It is interesting to note that the greater inaccuracy of H_s as compared with H_u in the tables is almost exactly what should be expected due to the factor 1.5 before $\Delta\tau_1$ in eq. (21), thus indicating that the dispersion is almost entirely due to this kind of errors.

As $\Delta\tau_1$ and $\Delta\tau_2$ enter into $\Delta_\sigma H$ with opposite signs, systematic errors influencing τ_1 and τ_2 in the same manner will to a certain extend compensate each other and therefore be of less importance. For instance will an unforeseen minor deviation from the exact time of the timepiece used in controlling the series of observations not necessarily seriously falsify the results.

Errors of this type will most probably occur when τ_1 and τ_2 are not too far separated in time. When in table IV excluding the data from the 13th of February which seem to have been encumbered with errors of another kind, the remaining σ-values treated in the usual way (see e.g. Fisher (18)) give a correlation coefficient $\varrho = 0.41$ with a significance represented by $P = 0.08$ (a probability of 0.08 that correlation does not exist). There is in other words some evidence of a day to day covariation of the σ-values that may be interpreted as due to systematic σ-errors. To get the full benefit of the counterbalancing effect one should therefore never combine a σ_1-value from any day with a σ_2-value from another in a height-computation. Nor should the mean of a series of height-observations be calculated from *mean* values of σ_1 and σ_2, especially not when the series contains incomplete σ-values sets (i.e. days with only one σ-value). The lastmentioned procedure would moreover mask possible seasonal variations of H_u and H_s.

A second source of error lies in incorrect reading of the zenithangles ζ. Partial derivation of H with respect to this angle (eq.s (11a) and (16a), eventually eq.s (9a) and (13d) combined with (11) and (16)) and application of the same data for ζ and σ as before, ultimately gives:

$$\begin{aligned} \Delta_{\zeta_1} H_u &= 0.5\,\Delta\zeta_1 \text{ km} \\ \Delta_{\zeta_2} H_u &= -8.8\,\Delta\zeta_2 \text{ ,,} \\ \Delta_{\zeta_1} H_s &= 0.7\,\Delta\zeta_1 \text{ ,,} \\ \Delta_{\zeta_2} H_s &= -8.7\,\Delta\zeta_2 \text{ ,,} \end{aligned} \qquad (22)$$

As should be expected the adjustment towards zenith ($\zeta_1 = 0$) is far less critical than that towards $\zeta_2 = 80$. Systematic errors produced by incorrect arrangement of the measuring apparatus will therefore not cancel. It must be a claim that the elevation may be correctly read to at least $^1/_{10}$ of a degree lest the error produced in the heights shall surmount 1 km. It is further evidently necessary to take atmospheric refraction into account. Ignorance of refraction in the case dealt with here, would result in an erroneous increase of about 5 km in the computed heights.

Lastly is to be mentioned the errors originating from incorrect adjustment in the azimuthplane of the sun of the measuring apparatus. In § 4, 1 was stated that when the deviation (a) between the azimuthplanes of the sun and the direction of observation was less than $\pm 4°$, the errors produced in the heights were less than 0.3 km. Fig. 4 shows further that for deviations up to $\pm 7.5°$ the errors do not exceed 1 km. For deviations less than $\pm 2.5°$ the errors are practically zero.

This latter is especially fortunate because it makes it possible to undertake an observation without adjusting the apparatus during the exposure time if this does not exceed say 12 minutes. The azimuthplane of the collimator of the spectrograph must then be adjusted to coincide with the sun's azimuthplane in the middle of the exposure interval.

Inaccuracies in the σ- and ζ-determinations may produce positive or negative errors in H. When incorrect adjustment in the azimuthplane of the sun have been detected, the corrections to be applied to the heights computed by means of eq.s (11 a) or (16 a) are *always positive.*

This is important. In a great collection of heightcomputations one should expect the errors involved to be distributed at random. Those due to inaccuracies in σ and ζ should statistically neutralize each other since they are working both ways, whereas those due to ignorance of an existing a, will produce a one-sided lowering of the mean. The total correction to be applied to the mean values of the computed heights ought therefore to be positive.

The discussion in this and the preceeding section may be summarized as follows:

Application of the formulae (11 a) and (16 a) to observations carried out in direction with zenith angles $\zeta_1 = 0$, $\zeta_2 = 80°$, will give H_u and H_s correct within 2.5 km in single computations, if $|\Delta\tau_1 - \Delta\tau_2|$, $|\Delta\zeta|$, and $|a|$ do not exceed the values given in table V.

As previously pointed out the error corresponding to a given $\Delta\tau$ varies during the year. The two numbers in each latitude column of the $\Delta\tau$-row of table V give the values of the difference $|\Delta\tau_1 - \Delta\tau_2|$ which in the most unfavourable (first number) and most favourable (second number) season will produce a maximum error of 1.5 km error in H (corresponding to $\Delta\sigma \leqslant 0.05°$). It is difficult to give exact data for the second number at 60° and 70° N lat. Those given should apply for the last but one day in the springtime when observations are yet possible (see fig. 5).

When now returning to the results obtained by Vegard et al. no real discrepancy can be said to exist between the three series of observations contained in table III and IV. The determinations in table III are certainly the more inaccurate, probably due both to poor fixation of τ (see above) and not quite good adjustment

Table V.

Geogr. lat.	45°	60°	70°	Partiel max. error
$\lvert\Delta\tau_1 - \Delta\tau_2\rvert$	$^1/_3$—$^1/_2$ min.	$^1/_2$—10 min.	$^3/_4$—10 min.	1.5 km
$\lvert\Delta\zeta_1\rvert$		0.1°		0 ,,
$\lvert\Delta\zeta_2\rvert$		0.1°		1 ,,
$\lvert a\rvert$		2.5°		0 ,,
			Total max. error	2.5 km

of the zenithangle of the spectrograph. The more numerous height-data in table IV show far less dispersion around the means. This dispersion could be accounted for by a random error in τ of 1 to two minutes. There is further no indication of any grave error due to uncertainty in the ζ-determination, perhaps with one exception (data from 13/2—43). Errors due to incorrect adjustment in the azimuth-plane of the sun are not detectable. If present they should erroneously have lowered the means.

As to the value of H_s and H_u given by Dufay (table III) it is not possible, by the published data, to judge of their correctness. (The investigations have, however, probably been carried out in the South of France, so that errors due to uncertainty in τ (see fig. 5) will weigh more than in the Oslo-material.)

The value of H_s is of special interest. To bring the means, 42.5 km, from table IV down to the value 25 km given by Dufay, is required either a systematic error in the τ-determination of about 6 minutes or a systematic incorrectness of the ζ-adjustment of more than 2°. Neither of these contingencies seems probable. It should therefore be a fair assumption that the effective screening height is a little over 40 km.

§ 7. ON IMPROVING THE MANNER OF OBSERVING.

In the preceeding section were discussed in some detail the results and the precision of the mode of observing used by Vegard and collaborators. It was shown how critically the results depend on accurate measurements of σ and ζ and that the best series of heightcomputations as yet published show a dispersion of $\pm$ 6 km to $\pm$ 10 km around the means.

The question now arises if some other choice of directions of observations might increase the accuracy of the method. When planning a series of investigation due attention must be paid to the subsequent computation work. It is, therefore, out of question to observe in quite irregular fashion (involving the use of the complicated formulae (9) and (13)). Some specialisation with the direction of observation lying in the azimuth-plane of the sun seems the rational way of proceeding. Small corrections may in special cases if necessary be applied.

Now evaluation from eq s (9 d), (11), (13 a), and (16) in the way earlier adopted of the errors in H_u and H_s caused by an incorrectness of ζ gives:

$$\left.\begin{aligned} \Delta_{\zeta_1} H_s &= 111\,\frac{(\sin\zeta_2 - \sin(\sigma_1 - \sigma_2 + \zeta_2))\left(1 - \dfrac{\sin^2\zeta_2}{q_u^2}\right)^{\frac{1}{2}}\Delta\zeta_1{}^\circ\ \text{km}}{\sin^2(\sigma_1 - \zeta_1 - \sigma_2 + \zeta_2)} \\ \Delta_{\zeta_1} H_s &= 111\,\frac{(\sin\zeta_2 - \sin(\sigma_1 - \sigma_2 + \zeta_2))\sin(\sigma_2 - \zeta_2)}{\sin^2(\sigma_1 - \zeta_1 - \sigma_2 + \zeta_2)}\,\Delta\zeta_1{}^\circ\ \text{km} \end{aligned}\right\} \quad (23)$$

The expressions for $\Delta_{\zeta_2} H$ are obtained from eq. (23) by interchanging the indices.

$\dfrac{\Delta_\zeta H}{\Delta\zeta}$ has a minimum for a value of $\zeta = \zeta_m$ found from the equations:

$$\left.\begin{aligned} \left(\frac{d\sigma}{d\zeta}\frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\zeta}\right)\left(\frac{\Delta_\zeta H}{\Delta\zeta}\right) &= 0 \\ \left(\frac{d\sigma}{d\zeta}\frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\zeta}\right) q &= 0 \end{aligned}\right\} \quad (24)$$

The result is e.g.:

$$\zeta_{2m} = \frac{\zeta_1}{2} + \frac{\sigma_2 - \sigma_1}{2} \pm \frac{\pi}{4} \quad (25)$$

(There are more values of ζ_2 which make $\dfrac{\Delta_{\zeta_2} H}{\Delta\zeta_2}$ an extremum, but those given in eq. (25) are the only ones of practical interest.)

To every given value of ζ_1 there then exists a value of ζ_2 (and vice versa) for which the uncertainty of H will be the least possible.

For numerical evaluation of these minimum values knowledge of corresponding ζ- and σ-values is needed. The $\sigma(\zeta)$-function is defined by e.g. eq. (9 d) (q_u being constant), and two sets of corresponding ζ- and σ-values suffice to determine it numerically.

The curve in fig. 6 is drawn in accordance with the mean value-sets in table IV. (If these

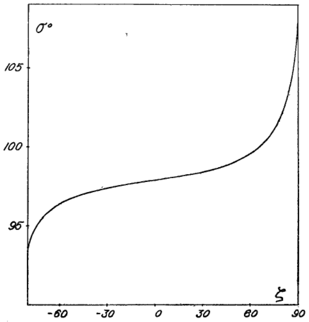

Fig. 6.

later on should prove to be somewhat incorrect, this will have no fundamental influence on this discussion.

Computations of some minimum values of $\frac{\Delta_{\zeta_2} H}{\Delta \zeta_2}$ and the values of $\frac{\Delta_{\zeta_1} H}{\Delta_{\zeta_1}}$ corresponding to the same sets og ζ_1 and ζ_2 are contained in table VI. The second column contains in addition the coefficients taken from eq.s (22).

A comparison of second and third column ($\zeta = 0$) in this table shows that the error in H caused by a given uncertainty in ξ_2 is reduced by a factor about 2,5 when ζ_2 is altered from 80° to 45.9°. On the other hand the errors caused by incorrectness in ξ_1 have become 3 to 4 times greater. The ζ_2-error may be reduced still more when ζ_1 is given numerically increasing negative values. But then soon the ζ_1-error will predominate and no total gain is obtained.

The uncertainty in ζ_1 and ζ_2 will presumeably be of the same magnitude. The maximum error in H will therefore be proportional to $S = \left|\frac{\Delta_{\zeta_1} H}{\Delta_{\zeta_1}}\right| + \left|\frac{\Delta_{\zeta_2} H}{\Delta_{\zeta_2}}\right|$. The sum rows in table VI show how S, with the special pairs of zenith-angles there applied, drops to a minimum for ζ_1 somewhere around — 45°. A more general oversight of the variation of S gives fig. 7 where the sum for several negative values of ζ_1 is plotted against ζ_2. The lowest curve is that for $\zeta_1 = -45$, and it has a minimum near $\zeta_2 = 45°$.

If the difficulty in obtaining good sodium-line spectra in directions with negative zenithangles (eastwards in the evening) could be overcome, pairs of observational series undertaken in the

Table VI.

Minium Values of $\frac{\Delta \zeta_2 H}{\Delta \zeta_2}$ *and Corresponding Values of* $\frac{\Delta \zeta_1 H}{\Delta \zeta_1}$.

ζ_1	0°	0	— 10	— 20	— 30	— 40	— 45	— 50	— 60	— 70
ζ_{2m}	(80)	45.5	40.5	35.5	30.6	25.6	23.3	20.7	15.9	11.3
$\frac{\Delta_{\zeta_1} H_u}{\Delta_{\zeta_1}}$	(0.5)	1.8	1.8	1.9	2.1	2.4	2.5	2.8	3.6	5.0
$\left(\frac{\Delta_{\zeta_2} H_u}{\Delta_{\zeta_2}}\right)_m$	(8.8)	3.6	3.1	2.5	2.1	1.7	1.5	1.3	1.0	0.7
Sum S_u	(9.3)	5.4	4.9	4.4	4.2	4.1	4.0	4.1	4.6	5.7
$\frac{\Delta_{\zeta_1} H_s}{\Delta_{\zeta_1}}$	(0.7)	2.0	2.0	2,1	2.2	2.5	2.6	2.9	3.6	5.1
$\left(\frac{\Delta_{\zeta_2} H_s}{\Delta_{\zeta_2}}\right)_m$	(8.7)	3.6	3.1	2.4	1.9	1.5	1.3	1.1	0.8	0.5
Sum S_s	(9.4)	5.6	5.1	4.5	4.1	4.0	3.9	4.0	4.4	5.6

sun's azimuthplane with zenith distances $-\zeta_1 = \zeta_2 = 45°$, should be very advantageous. Apart from having a simple set of formulae for computation ((9 e) and (13 b) still somewhat simplified) the maximum error in H caused by uncertainty in the determination of the zenith-angles is reduced to a minimum. (The absolute minimum at $-\zeta_1 = \zeta_2 = \frac{\pi}{4} + \frac{\sigma_2 - \sigma_1}{4} \approx 45.5°$ differs practically nothing from the value of S at 45°.) The gain as compared to the set of observations: $\zeta_1 = 0$, $\zeta_2 = 80°$ is about one to three.

Further, the ζ_1- and ζ_2-errors per degree will now be numerically equal. One-sided systematic errors in the adjustment of the spectrograph appearing in both series (ζ_1- and ζ_2-direction) will therefore to a certain extent counterbalance each other.

On the whole pairs of observational series with numerical equal zenith angles between 30° to 60° will have much the same qualifications and give satisfactory results. Convenient computation formulae and good precision may also be obtained with $\zeta_1 = 0$, $50° \leqslant \zeta_2 \leqslant 60°$ (see eq. (11 a), (16 a), and fig. 7).

While thus succeeding in reducing the ζ-errors by suitable choice of direction of observation, the effect on the σ-error by these operations will be to the contrary. According to eq. (20) which relates to directions $\zeta_1 = 0$, $\zeta_2 = 80°$ an uncertainty $\Delta\sigma$ of 0.04—0.05° will produce an error of about 1.5 km in H. A similar effect is produced by uncertainties in σ of 0.03° and 0.02° respectively for the observational manners $-\zeta_1 = \zeta_2 = 45°$ and $\zeta_1 = 0$, $\zeta_2 = 60°$. In wintertime at geographical latitudes 45°, 60° and 70° N this should correspond to an uncertainty $\Delta\tau$ (or rather $|\Delta\tau_1 - \Delta\tau_2|$ in the instance τ of disappearance of the enhanced sodiumlight of about 10, 15 and 20 seconds respectively (eq. (19) and fig. 5)). If determining τ from a time-intensity curve that precision should be attainable when the exposure intervals of the twilight spectra do not exceed $^3/_4$—$1^1/_2$ minute. At latitudes greater than say 55° N may in the favourable season (see end of § 5) intervals of 5—8 minutes be sufficiently accurate. This will, however, only apply for some few days every half year.

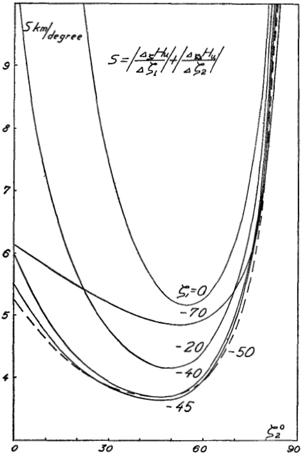

Fig. 7.

To obtain these short-exposure spectra a spectrograph of very high lightpower (say $f/0.7$ and coated for maximum transparency at 5890 Å) is required. When hypersensitised in some way, many ortho- or panchromatic photographic emulsions may successfully be used. Specially recommended are Eastman 103 a-T-plates.

Broad slit of the spectrograph together with properly orientated polaroid plates to filter out some part of the scattered sunlight may facilitate the intensity measurements. Although the polaroid filter will reduce the intensity of the sodiumlight with about 50 %, the continuous background is reduced considerably more thus improving the contrast.

Acknowledgment.

I am deeply indepted to Professor L. Vegard for the interest he has shown in discussing matters concerning this paper and for his most valuable advice.

SUMMARY.

1. A review is given of the results till now obtained from night sky observations of the position of the atmospheric sodium (§§ 1 and 2).
2. It is now a generally adopted opinion that the twilight effect of the radiating sodium is produced by some part of solar radiation for which the atmosphere below a certain height (H_s) acts as a screen. New formulae for the computation of H_s and of the upper limit H_u of the sodium which shows the twilight effect are calculated. The direction of observation need not coincide with the azimuthplane of the sun (§ 3).
3. The general formulae have been simplified by suitable choice of the direction of observation. In a couple of cases when simplification was impossible, good approximations are given (§ 4).
4. From the data given in the papers of Vegard et al. ((6), (13)) heights are recalculated in a manner slightly different from that previously adopted. Errors which might falsify the results are discussed. There is no indication of grave errors present in the material published by Vegard and Kvifte (13) (§ 6).
5. The conclusion is drawn that at present the most probable values of H_s and H_u are:

$$H_s = 42.5 \text{ km}$$
$$H_u = 104.5 \text{ km}$$

6. In order to suppress the errors likely to corrupt the results, the observations should be carried out:
   a. At a spot of high geographical latitude.
   b. During a further specified period in the late spring or early autumn.
   c. In the azimuthplane of the sun towards point in the sky with zenithdistances $-\zeta_1 = \zeta_2 = 45°$, or $\zeta_1 = 0$, $50° \leqslant \zeta_2 \leqslant 60°$.

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